MTH 203 - Quiz 2 Solutions

1. Let G be a group. If $G/Z(G)$ is cyclic, then show that G is abelian.

   Solution. Let $N = Z(G)$. Since G/N is abelian, we have $G/N = \langle xN \rangle$ for some $xN \in G/N$. In other words, for each $hN \in G/N$, there exists $k \in \mathbb{Z}$ such that $hN = x^k N$. Thus, every $g \in G$ can be written in the form $x^k z$ for some $z \in N$ and $k \in \mathbb{Z}$. Now consider arbitrary $g_1, g_2 \in G$. Then for $i = 1, 2$, we have $g_i = x^{k_i} z_i$, where $z_i \in N$ and $k_i \in \mathbb{Z}$. Thus, we have:

$$\begin{aligned} g_1 g_2 &= (x^{k_1} z_1)(x^{k_2} z_2) \\ &= (x^{k_1})(z_1 x^{k_2}) z_2 && \text{(Associativity in } G) \\ &= (x^{k_1})(x^{k_2} z_1) z_2 && \text{(Since } z_1 \in Z(G)) \\ &= (x^{k_1} x^{k_2})(z_1 z_2) && \text{(Associativity in } G) \\ &= (x^{k_1+k_2})(z_2 z_1) && \text{(Since } z_i \in Z(G)) \\ &= (x^{k_2+k_1})(z_2 z_1) \\ &= (x^{k_2} x^{k_1})(z_2 z_1) \\ &= x^{k_2}(x^{k_1} z_2) z_1 && \text{(Associativity in } G) \\ &= x^{k_2}(z_2 x^{k_1}) z_1 && \text{(Since } z_1 \in Z(G)) \\ &= (x^{k_2} z_2)(x^{k_1} z_1) && \text{(Associativity in } G) \\ &= g_2 g_1, \end{aligned}$$

   which shows that G is commutative.

2. Determine whether the following statements are true or false. Justify your answers. [5+5]

   (a) For $n \geq 3$, there exists a surjective homomorphism $\mathbb{Z}_{2n} \to D_{2n}$.

   (b) For $n \geq 3$, there exists an injective homomorphism $C_n \to D_{2n}$.

   Solution. (a) This statement is **false.** Suppose that there exists a surjective homomorphism $\varphi : \mathbb{Z}_{2n} \to D_{2n}$. Then by the First Isomorphism Theorem, we have:

$$\mathbb{Z}_{2n} / \ker \varphi \cong D_{2n}.$$

   Moreover, $\ker \varphi$ is cyclic as it a (normal) subgroup of the cyclic group $\mathbb{Z}_{2n}$. Thus, it follows that $\mathbb{Z}_{2n} / \ker \varphi$ is abelian as it is the quotient two abelian groups (why?). Since D_{2n} is non-abelian, this clearly contradicts the fact that $\mathbb{Z}_{2n} / \ker \varphi \cong D_{2n}$.

   (b) The statement is **true.** For $0 \leq k \leq n-1$, consider the map

$$\varphi : C_n \to D_{2n} = \langle r, s \rangle : e^{i2\pi k/n} \overset{\varphi}{\mapsto} r^k.$$

The map φ is clearly well-defined. Furthermore, φ is a homomorphism since:

$$\begin{array}{rcll} \varphi(e^{i2\pi k_1/n}e^{i2\pi k_2/n}) & = & \varphi(e^{i2\pi(k_1/+k_2)/n}) & \\ & = & r^{k_1+k_2} & \text{(By definition of } \varphi) \\ & = & r^{k_1}r^{k_2} & \\ & = & \varphi(e^{i2\pi k_1/n})\varphi(e^{i2\pi k_2/n}) & \text{(By definition of } \varphi) \end{array}.$$

To see that φ is injective, it suffices to show that $\ker\varphi = \{1\}$. However, this follows from the argument that:

$$\begin{array}{rcll} \ker\,\varphi & = & \{g \in C_n : \varphi(g) = 1\} & \text{(By definition of } \ker\varphi) \\ & = & \{e^{i2\pi k/n} \in C_n : \varphi(e^{i2\pi k/n}) = 1\} & \\ & = & \{e^{i2\pi k/n} \in C_n : r^k = 1\} & \text{(By definition of } \varphi) \\ & = & \{e^{i2\pi k/n} \in C_n : k = 0\} & \\ & = & \{1\}. & \end{array}.$$